Student Name: **SOLUTIONS**

SCOTCH COLLEGE

Practise One

MATHEMATICS: SPECIALIST

Question/Answer Booklet – Section 2 – Calculator-assumed

Teacher's Name: ____________________

Time allowed for this paper

Section	Reading	Working
~~**Calculator-free**~~	~~5 minutes~~	~~50 minutes~~
Calculator-assumed	10 minutes	100 minutes

Materials required/recommended for this paper

Section Two (Calculator-assumed): 100 marks

To be provided by the supervisor
Section Two Question/Answer booklet Formula sheet

To be provided by the candidate
Standard items: pens, pencils, pencil sharpener, eraser, correction fluid/tape, ruler, highlighters

Special items: drawing instruments, templates, notes on two unfolded sheets of A4 paper, and up to three calculators satisfying the conditions set by the School Curriculum and Standards Authority for this course.

Important Note to candidates
No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised notes or other items of a non-personal nature in the examination room. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Instructions to candidates

1. **All** questions should be attempted.

2. Write your answers in the spaces provided in this Question/Answer Booklet. Spare pages are included at the end of this booklet. If you need to use the space to continue an answer, indicate in the original answer space where the answer is continued, i.e. give the page number. Fill in the number of the question(s) that you are continuing to answer at the top of the page.

3. **Show all your working clearly.** Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat an answer to any question, ensure that you cancel the answer you do not wish to have marked.

4. It is recommended that you **do not use pencil** except in diagrams.

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Section Two: Calculator-assumed **(100 Marks)**

This section has **twelve (12)** questions. Answer **all** questions. Write your answers in the space provided.

Working time: 100 minutes.

Question 8 **(6 marks)**

A cylindrical metal disc expands when heated.

The cross-section 'height' of the disc is one fifteenth of the radius of the disc – i.e. $h = \frac{r}{15}$.

Determine the radius of the disc when the rate of changes of the volume (***V***) of the disc with respect to temperature (***t***) – i.e. $\frac{dV}{dt}$, is the same as the rate of change of the Surface Area (***SA***) or the disc with respect to temperature (***t***) – i.e. $\frac{dSA}{dt}$.

Solution	
$V = \pi r^2 h \quad h = \frac{r}{15}$ $\therefore V = \frac{\pi r^3}{15}$ $\frac{dV}{dr} = \frac{3\pi r^2}{15} = \frac{\pi r^2}{5}$ $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = \frac{\pi r^2}{5} \times \frac{dr}{dt}$	$SA = 2\pi r^2 + 2\pi rh \quad h = \frac{r}{15}$ $= 2\pi r^2 + \frac{2\pi r^2}{15}$ $\therefore SA = \frac{32\pi r^2}{15}$ $\frac{dSA}{dr} = \frac{64\pi r}{15}$ $\frac{dSA}{dt} = \frac{dSA}{dr} \times \frac{dr}{dt} = \frac{64\pi r}{15} \times \frac{dr}{dt}$
required to find when $\frac{dV}{dt} = \frac{dSA}{dt}$ – i.e. $\frac{\pi r^2}{5} \times \frac{dr}{dt} = \frac{64\pi r}{15} \times \frac{dr}{dt}$ i.e. $\frac{\pi r^2}{5} = \frac{64\pi r}{15}$ $3\pi r^2 = 64\pi r$ $r = \frac{64}{3} = 21\tfrac{1}{3}$ cm	
Specific behaviours	
✓ determines $\frac{dV}{dr}$	✓ determines $\frac{dV}{dt}$
✓ determines $\frac{dSA}{dr}$	✓ determines $\frac{dSA}{dt}$
✓ equates	✓ solves for ***r***

1234567-8

Question 9 **(7 marks)**

(a) Many chemical reactions follow Wilhelmy's Law which states that the velocity of the reaction is proportional to the concentration of the reacting substance – i.e. if a is the initial concentration of the reagent and x is the amount transformed at time t then:

$$\frac{dx}{dt} = k\left(a - x\right), \quad 0 \le x \le a$$

(i) solve this differential equation and express x in terms of kt, for $t \ge 0$ given $a = 10$ and $x = 0$ when $t = 0$. [3]

Solution
$\frac{dx}{dt} = k\left(a - x\right)$ $\int \frac{1}{a - x}\ dx = k\int 1\ dt$ $kt + c = -\ln\left(a - x\right)$ $\ln\left(a - x\right) = -kt + c$ $a - x = e^{-kt+c}$ when $t = 0$, $x = 0$ & $a = 10$ thus $a = e^{c} = 10$ $\therefore x = 10 - 10e^{-kt}$
Specific behaviours
✓ separates the variables   ✓ integrates correctly   ✓ finds 'c' or e^{c}

(ii) determine k given $x = 4$ when $t = 2$ and hence re-write your expression for x in terms of just t. [2]

Solution
given $x = 4$ when $t = 2$ $4 = 10 - 10e^{-2k}$ $\frac{-6}{-10} = e^{-2k}$ $e^{2k} = \frac{10}{6}$ thus: $k = \frac{1}{2}\ln\left(\frac{5}{3}\right)$ or $k = 0 \cdot 2554$ $\therefore x = 10 - 10e^{-\frac{1}{2}\ln\left(\frac{5}{3}\right)t}$ or $x = 10 - 10e^{-0\cdot2554t}$
Specific behaviours
✓ determines the value of k – decimal or exact   ✓ re-writes expression for x

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Question 9 (a) *continued*

(iii) determine the amount of the substance left (un-transformed) after 5 minutes. [2]

Solution
when $t = 5$ $x = 10 - 10e^{-\frac{1}{2}\ln\left(\frac{5}{3}\right)\times 5}$ $x = 7 \cdot 21$ units hence amount left (un-transformed) = 10 – 7·21 = **2·79 units**
Specific behaviours
✓ calculates x ✓ calculates amount ***left !***

1234567-8

Question 10 **(8 marks)**

At noon, spy drone ***A*** takes off 20 km north of Peeko town site with a velocity vector of $5\underset{\sim}{i} - 16\underset{\sim}{j} + 0\cdot 4\underset{\sim}{k}$ km/h. Also at noon, spy drone ***B*** takes off 20 km east of Peeko town site with a velocity vector of $-16\underset{\sim}{i} + 5\underset{\sim}{j} + 0\cdot 4\underset{\sim}{k}$ km/h. The vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ are unit vectors in the directions East, North and vertically upwards respectively.

(a) Determine the position vectors of both spy drones, (***A*** and ***B***), 1 hour after take-off. [2]

Solution
$A = \begin{pmatrix} 0 \\ 20 \\ 0 \end{pmatrix} + \begin{pmatrix} 5 \\ -16 \\ 0\cdot 4 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 0\cdot 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 20 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} -16 \\ 5 \\ 0\cdot 4 \end{pmatrix} = \begin{pmatrix} 4 \\ 5 \\ 0\cdot 4 \end{pmatrix}$
Specific behaviours
✓ determines position vector for ***A*** ✓ determines position vector for ***B***

(b) How far apart are the spy drones after 1 hour? [1]

Solution
$AB = B - A = \begin{pmatrix} 4 \\ 5 \\ 0\cdot 4 \end{pmatrix} - \begin{pmatrix} 5 \\ 4 \\ 0\cdot 4 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ Distance apart = $\sqrt{1^2 + (-1)^2 + 0^2} = 1\cdot 41$ km
Specific behaviours
✓ calculates distance apart

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Question 10 *continued*

(c) (i) determine the minimum distance between the two spy drones and when this occurs. [3]

Solution
$A(t)=\begin{pmatrix}0\\20\\0\end{pmatrix}+t\begin{pmatrix}5\\-16\\0\cdot4\end{pmatrix}=\begin{pmatrix}5t\\20-16t\\0\cdot4t\end{pmatrix}$ $B(t)=\begin{pmatrix}20\\0\\0\end{pmatrix}+t\begin{pmatrix}-16\\5\\0\cdot4\end{pmatrix}=\begin{pmatrix}20-16t\\5t\\0\cdot4t\end{pmatrix}$ $BA(t)=A(t)-B(t)=\begin{pmatrix}5t\\20-16t\\0\cdot4t\end{pmatrix}-\begin{pmatrix}20-16t\\5t\\0\cdot4t\end{pmatrix}=\begin{pmatrix}21t-20\\20-21t\\0\end{pmatrix}$ minimum distance apart = $\lvert BA(t)\rvert=\sqrt{(21t-20)^2+(20-21t)^2}$ fmin (***CAS***): minimum distance = 0 km at 0·95 hours (12:57pm)
Specific behaviours
✓ formulates ***BA(t)***   ✓ determines the minimum distance   ✓ states ***when*** = 12:57 pm, the minimum distance occurs

(ii) interpret your result [1]

Solution
The spy drones **collide** at 12:57 pm
Specific behaviours
✓ states that the result indicates a collision

(d) In light of the result you obtained in part (c), how would you now interpret your answer in part (b)? [1]

Solution
The spy drones collide before 1 hour thus the answer in part (b) is not valid
Specific behaviours
✓ states that the answer for part (b) is not possible

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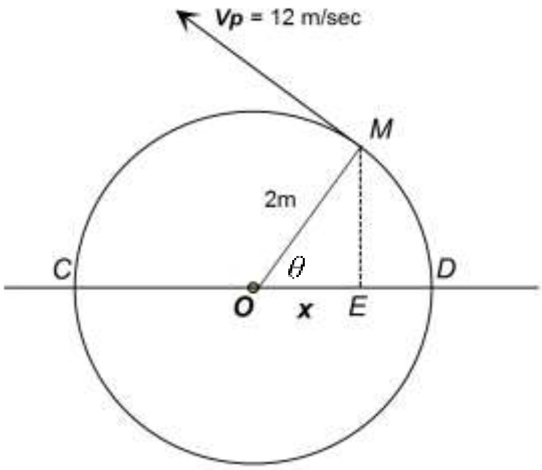
Question 11 **(7 marks)**

A mini-motorbike is attached to a metal stake at the centre of a circular track. The mini-motorbike (***M***) starts at the point ***D*** and moves on a circular path with a constant speed of 12 m/sec. The radius of the track is 2m.

The electric power-cell (point ***E***) is attached to the 'diameter' (***CD***) – the *x*-axis. The electric power-cell travels from ***D*** to ***C*** and then from ***C*** back to ***D*** as the mini-motorbike completes one lap.

The displacement of the electric power-cell (point ***E***) from the centre stake (point ***O***) is represented by ***x***.



(a) determine:

(i) the value of θ after 1 second and hence write an equation for θ in terms of time t. [2]
(i) an expression for ***x*** – the displacement of ***E*** with respect to time (***t***). [1]
(ii) an expression for ***v*** – the velocity of ***E*** with respect to time (***t***). [1]
(iii) an expression for ***a*** – the acceleration of ***E*** with respect to time (***t***). [1]

Solutions
In one second, distance travelled (arc length) = 12m $a = r\theta$ $12 = 2\theta$ $\therefore\ \theta = 6$ $\theta = 6t$ rad/sec $\therefore x = 2\cos 6t$ $v = \dfrac{dx}{dt} = -12\sin 6t$ $a = \dfrac{dv}{dt} = -72\cos 6t$
Specific behaviours
✓ calculates the value of θ ✓ states an equation for θ in terms of t ✓ writes the correct expression for displacement ***x***   ✓ writes the correct expression for velocity ***v***   ✓ writes the correct expression for acceleration ***a***

(b) hence describe the motion of the point ***E***.
Use your results from above to justify your conclusion. [2]

Solution
$a = -72\cos 6t = -36(2\cos 6t) = -36x$ thus $\ddot{x} = a = -6^2 x$ which is of the form $\ddot{x} = -k^2 x$ hence the motion of point ***A*** is Simple Harmonic Motion
Specific behaviours
✓ shows the relationship between ***a*** and ***x***   ✓ states that the motion is Simple Harmonic Motion

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Question 12 **(10 marks)**

(a) Find the square roots of $-15-8i$. [4]

Solution
$-15-8i = z = 17cis(-2\cdot 65 + 2n\pi)$
$\therefore \sqrt{-15-8i} = \sqrt{z} = (17)^{\frac{1}{2}} cis(-2\cdot 65... + 2n\pi)^{\frac{1}{2}} = \sqrt{17}cis(-1\cdot 325... + n\pi)$
for $n=0 \Rightarrow \sqrt{17}cis(-1\cdot 325...) = \sqrt{17}\left(\cos(-1\cdot 325...) + i\sin(-1\cdot 325...)\right)$
$\Rightarrow$ [CAS] − compToRect $\Rightarrow$ $\mathbf{1-4}\boldsymbol{i}$
for $n=1 \Rightarrow \sqrt{17}cis(-1\cdot 325... + \pi) = \sqrt{17}\left(\cos(-1\cdot 325... + \pi) + i\sin(-1\cdot 325... + \pi)\right)$
$= -\sqrt{17}\left(\cos(-1\cdot 325...) + i\sin(-1\cdot 325...)\right)$
$\Rightarrow$ [CAS] − compToRect $\Rightarrow$ $\mathbf{-1+4}\boldsymbol{i}$
see alternative methods at end of solutions
Specific behaviours
✓ turning rectangular form into ***CiS*** form
✓ determining square roots in general form $\rightarrow \sqrt{17}cis(-1\cdot 325... + n\pi)$
✓ determining two square roots
✓ expressing answers in rectangular form

(b) Suppose that z and w are complex numbers such that $z+w=1$ and $|z|=|w|$.

Show that $\text{Re}(z)=\text{Re}(w)=\frac{1}{2}$ and $\text{Im}(z)=-\text{Im}(w)$. [6]

Solution
Let $z=a+bi$ and $w=c+di$.
$\therefore\ z+w=a+bi+c+di=1$
$\therefore\ a+c=1$ and $b+d=0$
$\Rightarrow\ b=-d$
$\therefore\ \text{Im}(z)=-\text{Im}(w)$
$|z|=|w|$
$\therefore\ a^2+b^2=c^2+d^2$
$a^2+(-d)^2=c^2+d^2$
$\therefore\ a^2=c^2$
Since $a+c=1$, $a\neq -c$
$\therefore\ a=c=1/2$
$\therefore\ \text{Re}(z)=\text{Re}(w)=1/2$
Specific behaviours
✓ defines z and w and determines $z+w=1$
✓ equates Real and Imaginary parts
✓ demonstrates solution for $\text{Im}(z)=-\text{Im}(w)$
✓ determines magnitudes and uses substitution
✓ uses $a+c=1$ to eliminate $a=-c$
✓ demonstrates solution for $\text{Re}(z)=\text{Re}(w)=\frac{1}{2}$

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Question 12 (a)

Alternative methods of solution

Alternative Method 1:

$-15-8i \;=\; 17\{\cos(\theta+2k\pi)+i\sin(\theta+2k\pi)\}$ where $\cos\theta=-15/17$, $\sin\theta=-8/17$

Then the square roots of $-15-8i$ are: $\sqrt{17}\left(\cos\theta/2+i\sin\theta/2\right)$ (1)

and $\sqrt{17}\left(\cos(\theta/2+\pi)+i\sin(\theta/2+\pi)\right)=-\sqrt{17}\left(\cos\theta/2+i\sin\theta/2\right)$ (2)

Now $\cos\theta/2 \;=\; \pm\sqrt{(1+\cos\theta)/2} \;=\; \pm\sqrt{(1-15/17)/2} \;=\; \pm 1/\sqrt{17}$

$\sin\theta/2 \;=\; \pm\sqrt{(1-\cos\theta)/2} \;=\; \pm\sqrt{(1+15/17)/2} \;=\; \pm 4/\sqrt{17}$

since θ is an angle in the third quadrant, $\theta/2$ is an angle in the second quadrant.

Hence $\cos\theta/2=-1/\sqrt{17}$, $\sin\theta/2=4/\sqrt{17}$ and so from (1) and (2) above the required square roots are: $-1+4i$ and $1-4i$.

[as a check note that $(-1+4i)^2=(1-4i)^2=-15-8i$].

Alternative Method 2:

Let $p+iq$, where p and q are real, represent the required square roots.

Then $(p+iq)^2=p^2-q^2+2pqi=-15-8i$ or (3) $p^2-q^2=-15$, (4) $pq=-4$

Substituting $q=-4/p$ from (4) into (3), it becomes $p^2-16/p^2=-15$ or

$p^4+15p^2-16=0$, i.e. $(p^2+16)(p^2-1)=0$ or $p^2=16$, $p^2=1$.

Since p is real, $p=\pm 1$. From (4) if $p=1$, $q=-4$; if $p=-1$, $q=4$.

Thus the roots are: $-1+4i$ and $1-4i$.

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more ponderings regarding Q12 b).

Given: z and w complex numbers and $z+w=1$ and $|z|=|w|$

a student argued that:

If $|z|=|w|$ *given*

then $|z||w|=|w|^2$ *multiply both sides by* $|w|$

and since $|z||\overline{z}|=|z|^2$ *known property*

w must be the conjugate of z

Is this argument correct?

<u>Analysis</u>

Since it is given that $|z|=|w|$, then the logic used by the student supports the fact that $|w|=|\overline{z}|$ but no more since $|z|=|\overline{z}|$. The given info that $z+w=1$ has not been used in the written working by the student and is integral to their argument.

To show that $w=\overline{z}$, using both givens:

Consider letting $z=x+iy$, then $w=1-x-iy$ and $x^2+y^2=(1-x)^2+y^2$ $-\left(|z|=|w|\right)$.

Hence, $x^2=(1-x)^2$

thus either $x=+(1-x)$ i.e. $x=0\cdot5$ or $x=-(1-x)$ no solution

Therefore, $z=0\cdot5+\mathrm{k}i$, k a real number and $w=0\cdot5-\mathrm{k}i$, i.e. w is the conjugate of z .

<u>Further</u>

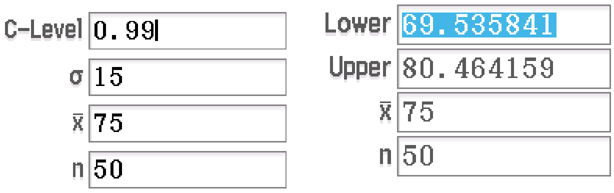
If $z=3+4i$ and $w=4+3i$, then $|z|=|w|$, but they are not conjugates of each other.
The additional information that $z+w=1$ is required to show that if both the conditions hold then they are conjugates of each other.
They clearly have opposite imaginary components since $w=1-z$ but without the modulus condition they do not necessarily have the same real component.

1234567-8

Question 13 **(10 marks)**

The adult length of a species of fish is known to be normally distributed with a mean length of 75cm and standard deviation of 15cm. It is suspected that a random sample of 50 adult fish of mean lengths of 80cm, belong to the same species.

(a) Determine at a 1% level of significance if this suspicion is true. (3)



$$69.54 \le \mu \le 80.46$$

Based on a 99% confidence interval, the mean of the sample is within the range and therefore suggests that the sample is from the same species.

(b) Determine the level of the significance between the mean length of this sample and the known length. (4)

$$m = \bar{x} + z \times \frac{s}{\sqrt{n}}$$

$$|m - \bar{x}| = z \times \frac{s}{\sqrt{n}}$$

$$5 = z \times \frac{15}{\sqrt{50}}$$

$$z = 2.357$$

$$Z \quad N(0,1^2) \quad P(-2.357 < z < 2.357) = 0.98148$$

$$(1 - 0.98148) \times 100 = 1.8\%$$

A new sample of n fish is going to be collected.

(c) Find n if the probability that the error between the sample mean and the true mean is no more than 4 cm is 0.97. (3)

Sample size: $n = \left(\frac{z \times \sigma}{d}\right)^2$ where d is the required value of the difference from the mean.

$$n = \left(\frac{z \times 15}{4}\right)^2$$

$$n = \left(\frac{2.17 \times 15}{4}\right)^2$$

$$n = 66.23$$

$$n = 67$$

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Question 14 **(14 marks)**

The number of mobile phones, N, owned in a certain community after t years, may be modelled by $\ln(N) = 6 - 3e^{-0.4t}, \quad t \geq 0$

(a) Verify by substitution that $\ln(N) = 6 - 3e^{-0.4t}$ satisfies the differential equation

$$\frac{1}{N}\frac{dN}{dt} + 0.4\ln(N) - 2.4 = 0$$

(3)

$$\ln(N) = 6 - 3e^{-0.4t}$$
$$\frac{1}{N}dN = 1.2e^{-0.4t}dt$$
$$\frac{1}{N}\frac{dN}{dt} = 1.2e^{-0.4t}$$

$$LHS =$$
$$= \frac{1}{N}\frac{dN}{dt} + 0.4\ln(N) - 2.4$$
$$= 1.2e^{-0.4t} + 0.4(6 - 3e^{-0.4t}) - 2.4$$
$$= 1.2e^{-0.4t} 2.4 - 1.2e^{-0.4t} - 2.4$$
$$= 0$$
$$= RHS$$

(b) Find the initial number of phones owned in the community. (1)

$$\ln(N) = 6 - 3e^{-0.4t}, \quad t \geq 0$$
$$\ln(N) = 6 - 3e^{0}$$
$$\ln(N) = 3$$
$$N = e^{3}$$
$$N = 20.09$$
$$N = 20$$

(c) Find the limit $\ln(N)$ as t approaches infinity. (1)

$$\ln(N) = 6 - 3e^{-\infty}$$
$$\ln(N) = 6$$

1234567-8

Question 14 *continued*

(d) Using the mathematical model, find the limiting number of mobile phones that would eventually be owned in the community.

(2)

$$\ln(N) = 6 - 3e^{-\infty}$$
$$\ln(N) = 6$$
$$N = e^{6}$$
$$N = 403$$

The maximum phones that the company will eventually own is 403.

The differential equation in part (a) can also be written in the form $\frac{dN}{dt} = 0.4N(6 - \ln(N))$

(e) Find $\frac{d^2N}{dt^2}$ in terms of N and ln(N)

(3)

$$\frac{d^2N}{dt^2} = \frac{d\left(\frac{dN}{dt}\right)}{dN} \times \frac{dN}{dt}$$
$$\frac{d\left(\frac{dN}{dt}\right)}{dN} = 2.4 - (0.4\ln(N) + 0.4)$$
$$\frac{d\left(\frac{dN}{dt}\right)}{dN} = 2 - 0.4\ln(N)$$
$$\frac{d^2N}{dt^2} = (2 - 0.4\ln(N))(2.4N - 0.4N\ln(N))$$

Question 14 *continued*

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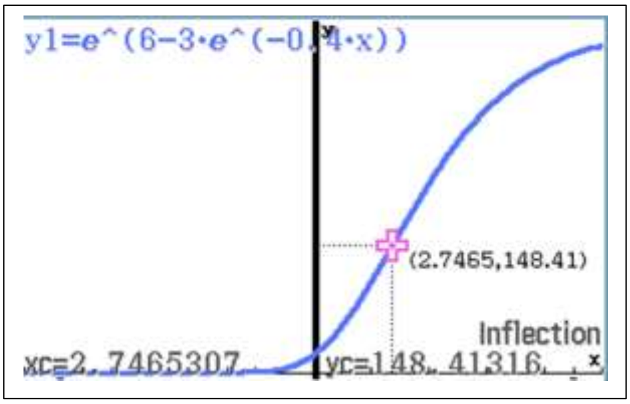
(f) The graph of N as a function of t has a point of inflection. Find the co-ordinates of the value of this point. Give the value of t correct to one decimal place and the value of N correct to the nearest integer. (2)

$\ln(N) = 6 - 3e^{-0.4t}$

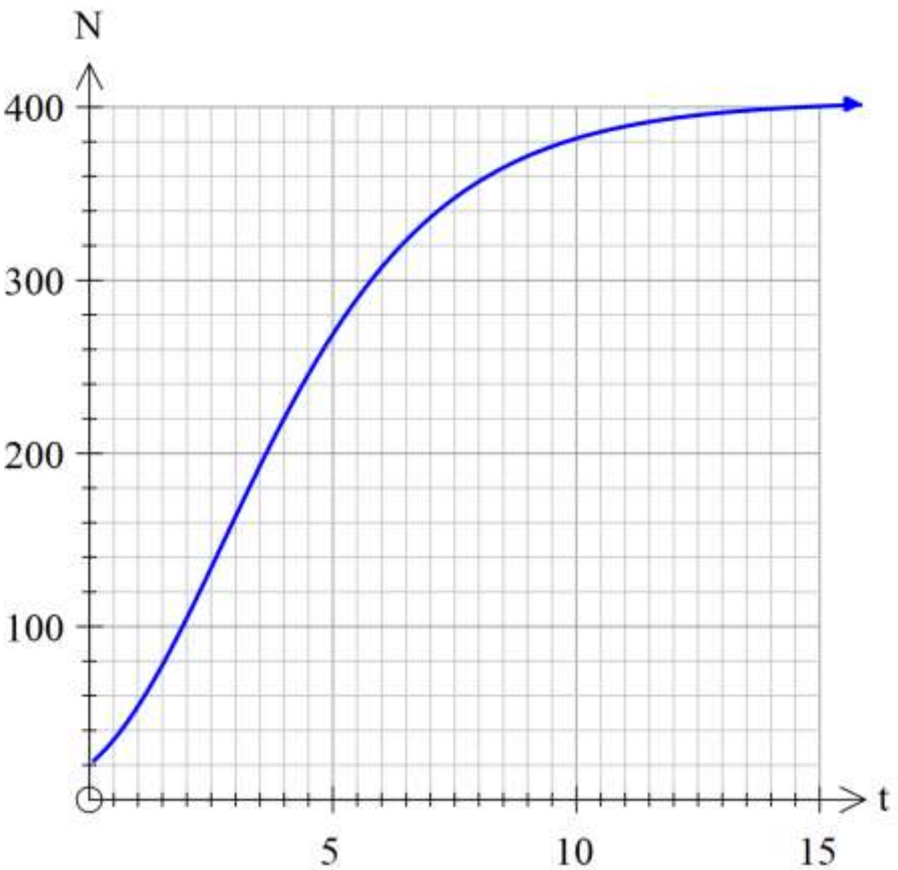
$N = e^{6-3e^{-0.4t}}$

$t = 2.8$

$N = 148$



(g) Sketch the graph of N as a function of t for $0 \le t \le 15$ (2)



Question 15 (13 marks)

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(a) Two vehicles leave a checkpoint at the same time. The first vehicle drives North at a speed of 50 km/h. The second vehicle travels at a speed of 65 km/h on a bearing of 060°.

(i) Let $\underset{\sim}{x}$ represent the position vector of the first vehicle at any time t and let $\underset{\sim}{y}$ represent the position vector of the second vehicle at any time t.
Determine expressions for $\underset{\sim}{x}$ and $\underset{\sim}{y}$. [2]

Solution

$\underset{\sim}{x} = 50t\,\underset{\sim}{j}$

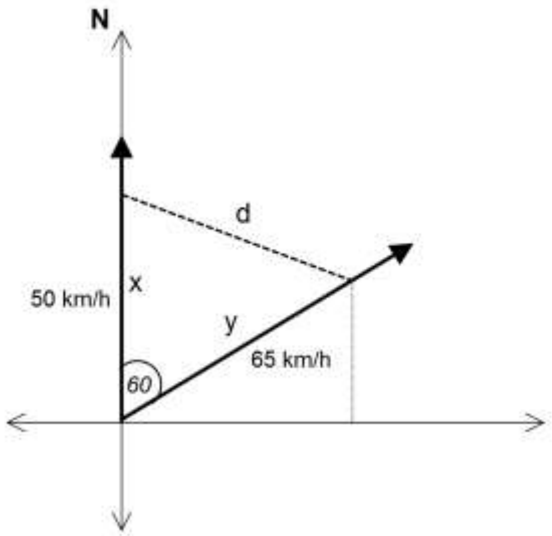
for $\underset{\sim}{y}$: $\underset{\sim}{i}$ component

$$\Rightarrow \cos 30° = \frac{\underset{\sim}{i}_{\text{component}}}{65} \Rightarrow \underset{\sim}{i}_{\text{component}} = \frac{65\sqrt{3}}{2}$$

for $\underset{\sim}{y}$: $\underset{\sim}{j}$ component

$$\Rightarrow \sin 30° = \frac{\underset{\sim}{j}_{\text{component}}}{65} \Rightarrow \underset{\sim}{j}_{\text{component}} = \frac{65}{2}$$

$$\therefore\ \underset{\sim}{y} = \frac{65\sqrt{3}}{2}t\,\underset{\sim}{i} + \frac{65}{2}t\,\underset{\sim}{j}$$



Specific behaviours

✓ correct $\underset{\sim}{x}$ position vector
✓ correct $\underset{\sim}{y}$ position vector

(ii) Hence, or otherwise, find the constant rate of change of the distance between the two vehicles and interpret your result. [3]

Solution

$$\underset{\sim}{d} = \left|\underset{\sim}{y} - \underset{\sim}{x}\right| = \left|\frac{65\sqrt{3}}{2}t\,\underset{\sim}{i} + \frac{65}{2}t\,\underset{\sim}{j} - 50t\,\underset{\sim}{j}\right|$$

$$= \left|\frac{65\sqrt{3}}{2}t\,\underset{\sim}{i} - \frac{35}{2}t\,\underset{\sim}{j}\right| = \sqrt{3475t^2} = \sqrt{3475}\ t$$

$$\therefore\ \frac{d}{dt}(d) = \sqrt{3475} \approx 58\cdot95 \text{ km/h}$$

the cars are moving away from each other at a constant speed of 58·95km/hr

Specific behaviours

✓ determines magnitude of $\underset{\sim}{d}$
✓ calculates rate
✓ interprets result

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Question 15 *continued*

(b) A third vehicle, travelling on a path $\underset{\sim}{r}_C = 4\cos(\pi t)i - 3\sin(\pi t)j$ leaves its starting point at the same time as the other two vehicles.

(i) Find the distance that vehicle C is from the other two vehicles at the start. [2]

$$\underset{\sim}{r}_C = 4\cos(0)i - 3\sin(0)j$$
$$\underset{\sim}{r}_C = 4i$$

4 km from the check point to the East.

(ii) Find the Cartesian equation that would represent the path of vehicle C. [2]

$$x = 4\cos(\pi t)$$
$$y = -3\sin(\pi t)$$
$$\left(\frac{x}{4}\right)^2 + \left(\frac{-y}{3}\right)^2 = \cos^2(\pi t) + \sin^2(\pi t)$$
$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

(iii) Find the maximum speed and the first time this occurs. [2]

The object is an ellipse travelling in a clockwise direction from (0,4)

$$v_C = -4\pi\sin(\pi t)i - 3\pi\cos(\pi t)j$$
$$|v| = \sqrt{16\pi^2\sin^2(\pi t) + 9\pi^2\cos^2(\pi t)}$$
$$|v| = \sqrt{16\pi^2\sin^2(\pi t) + 9\pi^2(1 - \sin^2(\pi t))}$$
$$|v| = \pi\sqrt{7\sin^2(\pi t) + 9}$$

$$\sin(\pi t) = \pm 1$$

Max when $\pi t = \frac{\pi}{2}, \frac{3\pi}{2}$

$$t = \frac{1}{2} \qquad v = 4\pi \ km/hr$$

The maximum speed is 12.6 km/hr and first occurs

1234567-8

(iii) Find the distance the vehicle C travels in the first 2 seconds.

[2]

$$|v| = \pi\sqrt{7\sin^2(\pi t)+9}$$

$$d = \int_0^2 \pi\sqrt{7\sin^2(\pi t)+9}\ dt$$

$$d = 22.10\,km$$

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Question 16 (6 marks)

(b) When a current I flows through an instrument the needle on the instrument rotates at an angle of θ $(\theta \neq 0)$ according to the function $I = \omega \tan\theta$, where ω is a constant. A small error $\delta\theta$ is made in reading the instrument.

(i) Show that the resulting fractional error in I is given by: $\dfrac{\delta I}{I} = \dfrac{2\delta\theta}{\sin 2\theta}$ [4]

Solution
$I = \omega\tan\theta$ so $\dfrac{dI}{d\theta} = \dfrac{\omega}{\cos^2\theta}$ $\delta I \approx \dfrac{dI}{d\theta}\cdot\delta\theta = \dfrac{\omega}{\cos^2\theta}\cdot\delta\theta$ $\dfrac{\delta I}{I} = \dfrac{\omega}{\cos^2\theta}\cdot\dfrac{1}{\omega\tan\theta}\cdot\delta\theta$ $= \dfrac{\omega}{\cos^2\theta}\cdot\dfrac{\cos\theta}{\omega\sin\theta}\cdot\delta\theta$ $= \dfrac{\delta\theta}{\sin\theta\cos\theta}$ $\therefore \dfrac{\delta I}{I} = \dfrac{2\delta\theta}{\sin 2\theta}$
Specific behaviours
✓ differentiates I w.r.t. θ ✓ determines δI ✓ establishes $\dfrac{\delta I}{I}$ ✓ simplifies $\dfrac{\delta I}{I}$ in required form

(ii) For a given $\delta\theta$, find the smallest positive value of θ that minimises the fractional error $\dfrac{\delta I}{I}$. [2]

Solution
The minimum value of $\dfrac{\delta I}{I}$ occurs when $\sin 2\theta$ is greatest $\Rightarrow \sin 2\theta = 1 \quad 2\theta = \dfrac{\pi}{2} \quad \therefore \theta = \dfrac{\pi}{4}$
Specific behaviours
✓ determines greatest when $\sin 2\theta = 1$ ✓ determines value of θ

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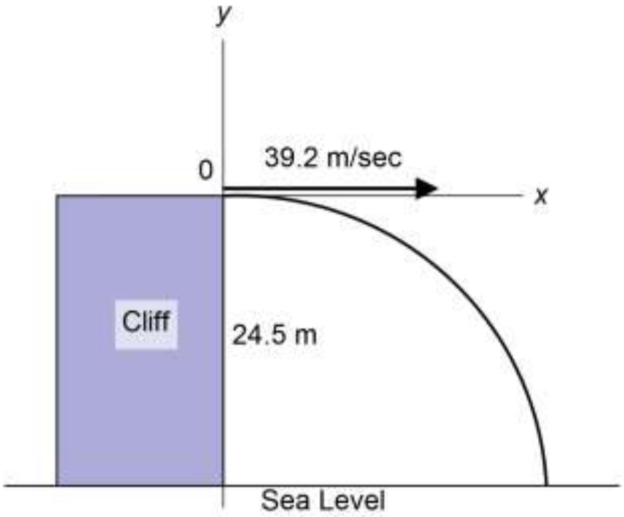
1234567-8

Question 17 **(6 marks)**

(a) A stone is projected horizontally, from the top of a cliff 24·5 m high. [3]

The *"initial"* or $t = 0$ conditions, when $x = 0$; and $y = 0$; are:

$$\frac{dx}{dt} = 39\cdot2 \text{ m/sec}; \quad \frac{dy}{dt} = 0 \text{ m/sec};$$
$$\frac{d^2x}{dt^2} = 0 \text{ m/sec}^2; \quad \frac{d^2y}{dt^2} = -9\cdot8 \text{ m/sec}^2$$



Determine the parametric equations of the path of the stone after t seconds.

Solution
$\frac{dx}{dt} = 39\cdot2 \text{ m/sec} \quad \Rightarrow \quad x = 39\cdot2t$ $\frac{d^2y}{dt^2} = -9\cdot8 \text{ m/sec}^2 \quad \Rightarrow \quad \frac{dy}{dt} = -9\cdot8t \text{ m/sec} \quad \Rightarrow \quad y = -4\cdot9t^2$
Specific behaviours
✓ determines parametric form of x ✓ determines $\frac{dy}{dt}$ ✓ determines parametric form of y

(b) Given that $y = \frac{4}{3}\pi x^3$, find the percentage change in y when x increases by 0·1% – i..e. $\delta x = 0\cdot001x$. [3]

Solution
$y = \frac{4}{3}\pi x^3 \quad \Rightarrow \quad \frac{dy}{dx} = 4\pi x^2$ $\frac{\delta y}{y} \approx \frac{1}{y}\frac{dy}{dx}\delta x \; = \; \frac{1}{\frac{4}{3}\pi x^3}\left(4\pi x^2\right)\left(0\cdot001x\right) \times 100\% \; = \; 0\cdot3\%$
Specific behaviours
✓ determines $\frac{dy}{dx}$ ✓ formulates $\frac{\delta y}{y}$ and substitutes known values   ✓ calculates percentage change in y

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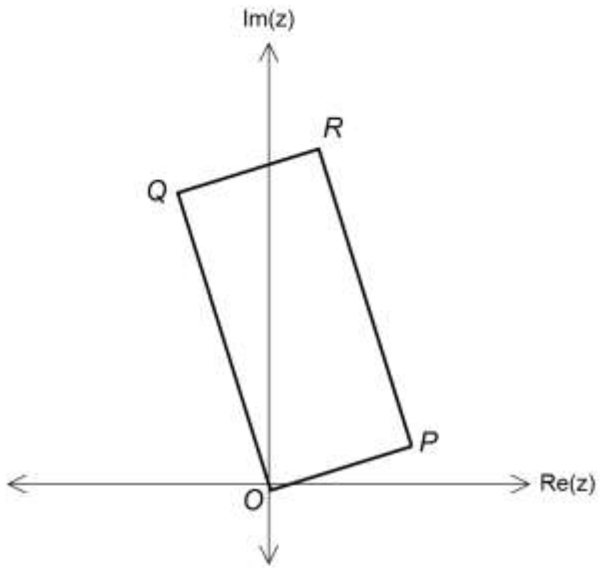
Question 18 **(7 marks)**

The points P, Q and R represent the complex numbers z, $2\mathrm{i}z$ and $(z+2\mathrm{i}z)$ respectively in an Argand diagram with origin O.

(a) Given that $z = x + iy$, where $x > 0$ and $y > 0$, sketch the Argand diagram. [3]

Solution
O is the origin, P represents $z = x + iy$ Q represents $2\mathrm{i}z$ thus OP is perpendicular to OQ N.B. $2\mathrm{i}z = 2i(x+iy) = 2ix - 2y = -2y + 2ix$. R represents $z + 2\mathrm{i}z$ i.e. $OP + OQ$ thus OR is the vector sum of OP and OQ.



Specific behaviours
✓ orients P with respect to the origin O ✓ correctly locates Q ✓ correctly locates R

(b) Deduce the value of $\angle POQ$ and the geometrical relationship between the points O, P, Q and R. [1]

Solution
$P = z$ and $Q = 2\mathrm{i}z$ thus $\angle POQ = 90°$
Specific behaviours
✓ correctly deduces that $\angle POQ = 90°$

1234567-8

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Question 18 *continued*

(c) Show that, if $y = 2x$, the point representing z^2 is collinear with the points O and R. [3]

Solution
$y = 2x$ $\Rightarrow z = x + iy = x + i2x$ $\overrightarrow{OR} = z + 2iz$ $\Rightarrow z + 2iz = (x + i2x) + 2i(x + i2x)$ $= x + i2x + i2x - 4x$ $= -3x + i4x = x(-3 + 4i)$ Let S = point representing z^2 $z^2 = (x + i2x)^2$ $= x^2 + i4x^2 - 4x^2$ $= -3x^2 + i4x^2$ $= x \times x(-3 + i4)$ $= x \times (z + 2iz)$ i.e. z^2 is a multiple of $(z + 2\text{i}\, z)$ $\Rightarrow$ S representing z^2 is collinear with the points O and R representing $(z + 2\text{i}\, z)$.
Specific behaviours
✓ establishes $z + 2iz = x(-3 + 4i)$ ✓ determines z^2 is a multiple of $(z + 2\text{i}\, z)$ ✓ concludes that z^2 is collinear with the points O and R

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Question 19 **(6 marks)**

I A sample of 30 is chosen from a binomial distribution with $p = 0.7$ and $n = 20$ and a graph of the sample is drawn.

II 20 samples of 30 are chosen from a binomial distribution with $p = 0.7$ and $n = 20$ and a graph of the means of each sample is drawn.

III 20 samples of 80 are chosen from a binomial distribution with $p = 0.7$ and $n = 20$ and a graph of the means of each sample is drawn.

How will the graphs from parts **I**, **II** and **III** be likely to compare. You should include the mean and standard deviation of each graph in your answers?

$$\mu = np \qquad \sigma = \sqrt{npq}$$

$$\mu = 20(0.7)$$
$$\mu = 14$$
$$\sigma = \frac{\sqrt{20(0.7)(0.3)}}{\sqrt{30}}$$
$$\sigma = \frac{\sqrt{4.2}}{\sqrt{30}}$$
$$\sigma = 0.374$$

$$\mu = 14$$
$$\sigma = \frac{\sqrt{20(0.7)(0.3)}}{\sqrt{30}}$$
$$\sigma = \frac{\sqrt{4.2}}{\sqrt{30}}$$
$$\sigma = 0.374$$

$$\mu = 14$$
$$\sigma = \frac{\sqrt{4.2}}{\sqrt{80}}$$
$$\sigma = 0.229$$

The graph of **I** should be skewed left with a mean of 14 and standard deviation of 0.374.

The graph of **II** should be symmetrical, normally distributed with a mean of 14 and standard deviation of 0.374.

The graph of **II** should be symmetrical, normally distributed with a mean of 14 and standard deviation of 0.229.

1234567-8

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